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Air Force Systems Command
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Dr. M. Guelman

September 1974

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THE CLOSED FORM SOLUTION OF TRUE PROPORTIONAL NAVIGATION

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20. Abstract The closed form solution of the equations of motion of an ideal missile pursuing a non-maneuvering target according to the true proportional navigation (TPN) law is obtained. In this case, commanded accelerations are applied in a direction normal to the interceptor-target line of sight. A linearized analysis is applied to study the TPN and a circle is found where capture can be demonstrated. A major point of this study is that the analysis of the closed form solution of TPN enables one to demonstrate the basic differences existing between the two most utilized forms of proportioned navigation, namely that pure proportional navigation has a much different capture envelope than the circle described for TPN.

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THE CLOSED FORM SOLUTION OF TRUE PROPORTIONAL NAVIGATION

ABSTRACT

The closed form solution is obtained of the equations of motion of an ideal missile pursuing a nonmaneuvering target according to the true proportional navigation law. An analysis of the solution is performed and the conditions for the missile to reach the target are determined.

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1. INTRODUCTION

Most modern air-to-air and surface-to-air missile systems use a form of proportional navigation in the homing phase of flight.

In proportional navigation control accelerations are generated proportional to the measured rate of rotation of the interceptor-target line of sight.

In the literature two basic forms of proportional navigation have been considered. These two forms are generally labelled, pure proportional navigation (PPN) |2| and true proportional navigation (TPN) |1|. In PPN commanded accelerations are applied normal to the missile velocity. In TPN commanded accelerations are applied in a direction normal to the interceptor-target line of sight. In both cases no closed-form solution is available, and linearised analysis was applied to study these two forms of proportional navigation.

Applying qualitative methods for the analysis of PPN |2| it was demonstrated that, provided a set of conditions relating the ratio of velocities and the constant of navigation, are fulfilled, capture can be assured for any initial conditions excepted for a precisely defined particular case.

In this study we determine the closed form solution of the differential equations describing the trajectories of a missile pursuing a nonmaneuvering target according to the true proportional navigation law. The solution was analised and it is demonstrated that capture is restricted for the cases where the initial conditions belong to a determined circle, defined as the circle of capture. In particular it can be shown that even if the missile is initially approaching the target there exists an entire region of initial conditions where capture cannot be assured. This strongly differentiates the two forms of proportional navigation as opposed to previous linearised analysis where equivalent results were obtained for both cases |1|, |3|.

An analysis is also made of the behaviour of the rate of rotation of the line of sight for the case where capture is assured. New results relating to the boundedness of the LOS rate are demonstrated.

2. PROBLEM STATEMENT

Consider a target T and missile M as points in a plane moving with velocities V_T and V_M respectively as shown in Fig. 1. The system can be described in a relative system of coordinates with its center at T and axis Tx along the straight line trajectory of the target.

In true proportional navigation |1| (TPN) the missile acceleration commanded, a_M, is applied normal to the line of sight, as opposed to pure proportional navigation |2| (PPN) where the missile acceleration commanded is applied normal to the missile velocity, as depicted in Fig. 2.

The equations of motion of the missile are derived in the following form:
Letting a dot denote differentiation with respect to time, the components of the relative velocity from missile to target are, in polar coordinates

$$V_r = \dot{r} = V_M \cos\alpha - V_T \cos\theta \tag{1}$$

$$V_\theta = r\dot{\theta} = V_M \sin\alpha + V_T \sin\theta \tag{2}$$

In proportional navigation the interceptor acceleration is proportional to the line of sight angular rate

$$a_M = c\dot{\theta} \tag{3}$$

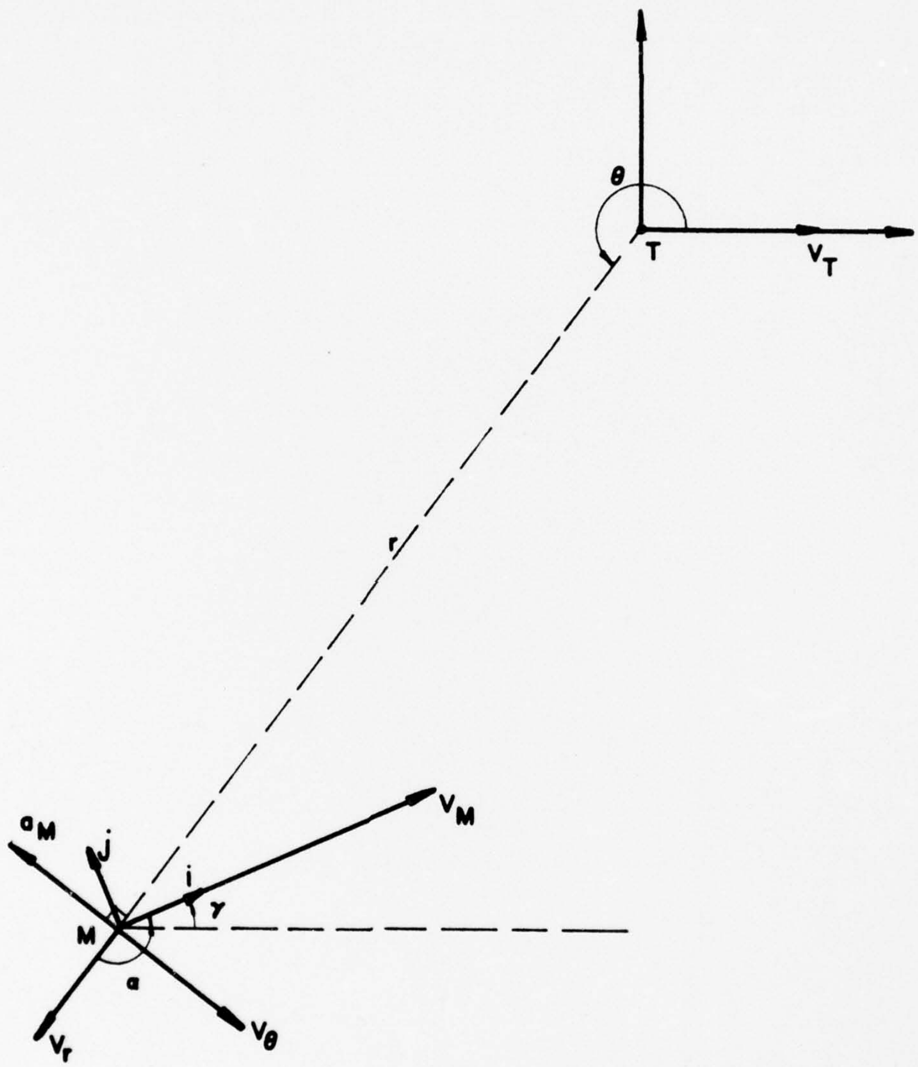

Fig. 1: Planar pursuit, true proportional navigation

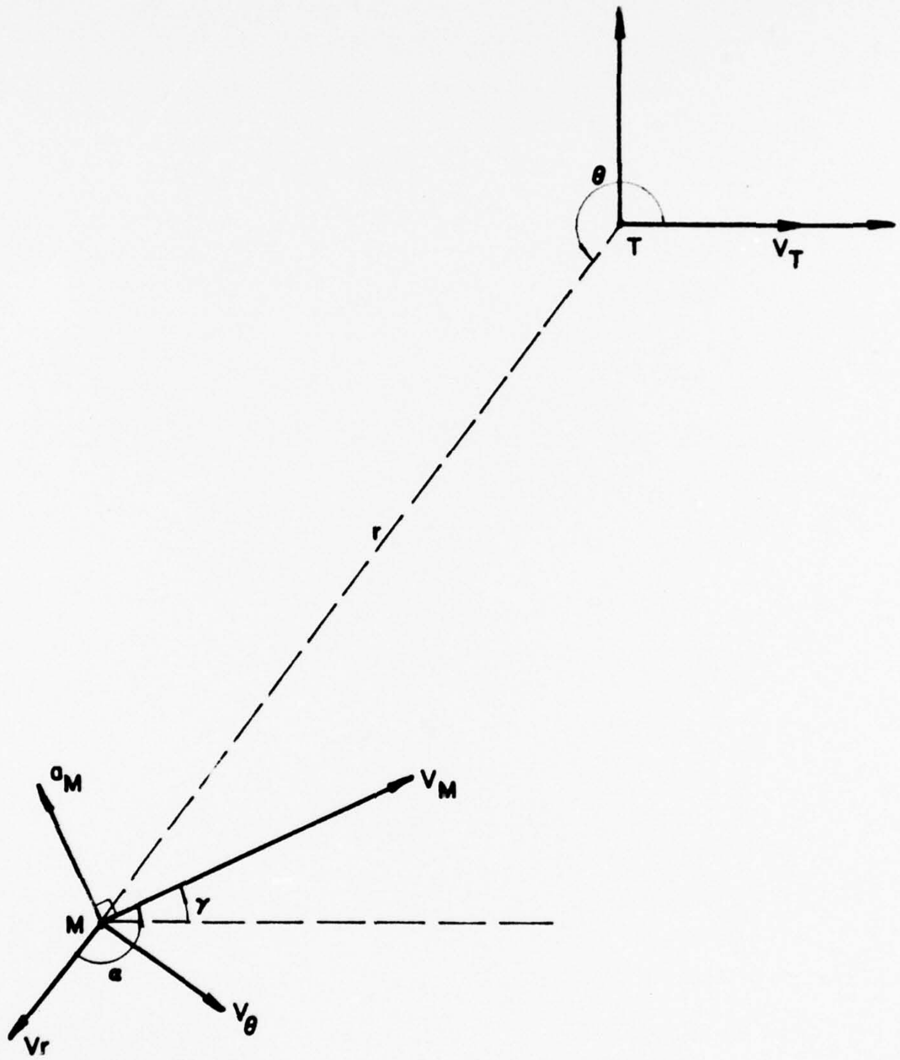

Fig. 2: Planar pursuit, pure proportional navigation

where in TPN c is generally defined as |1|

$$c = -\lambda V_{r_0} \tag{4}$$

with λ the navigation constant.

From the kinematics of a point

$$\vec{a}_M = \dot{\vec{V}}_M + \vec{\omega} \times \vec{V}_M \tag{5}$$

where

$$\vec{\omega} = \dot{\gamma} \vec{k} \tag{6}$$

is the angular velocity of the missile system of coordinates (i, j, k) with respect to an inertial reference.

Developping (5) for the TPN case,

$$\dot{\gamma} = -\frac{a_M \cos\alpha}{V_M} \tag{7}$$

$$\dot{V}_M = -a_M \sin\alpha \tag{8}$$

From Fig. 1

$$\gamma = \alpha + \theta - 2\pi \tag{9}$$

Differentiating (9) with respect to time and rearranging

$$\dot{\alpha} = \dot{\gamma} - \dot{\theta} \tag{10}$$

Replacing $\dot{\gamma}$ from (7) into (10)

$$\dot{\alpha} = -\frac{a_M \cos\alpha}{V_M} - \dot{\theta} \tag{11}$$

Differentiating (1) and (2) with respect to time

$$\dot{V}_r = \dot{V}_M \cos\alpha - V_M \sin\alpha\,\dot{\alpha} + V_T \sin\theta\,\dot{\theta} \tag{12}$$

$$\dot{V}_\theta = \dot{V}_M \sin\alpha + V_M \cos\alpha\,\dot{\alpha} + V_T \cos\theta\,\dot{\theta} \tag{13}$$

Replacing $\dot{V}_M$ and $\dot{\alpha}$ from (8) and (11) respectively into (12) and (13)

$$\dot{V}_r = (V_M \sin\alpha + V_T \sin\theta)\,\dot{\theta} \tag{14}$$

$$\dot{V}_\theta = -a_M - (V_M \cos\alpha - V_T \cos\theta)\,\dot{\theta} \tag{15}$$

Introducing in the right hand terms of (14) and (15) V_r and V_θ instead of their values as defined in (1) and (2)

$$\dot{V}_r = V_\theta\,\dot{\theta} \tag{16}$$

$$\dot{V}_\theta = -a_M - V_r\,\dot{\theta} \tag{17}$$

Finally replacing V_r by $\dot{r}$ and V_θ by $r\dot{\theta}$ in (16) and (17) and rearranging

$$\ddot{r} - r\dot{\theta}^2 = 0 \tag{18}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -a_M \tag{19}$$

Replacing a_M from (3) into (19) and rewritting (18)

$$\ddot{r} - r\dot{\theta}^2 = 0 \tag{20}$$

$$r\ddot{\theta} + (2\dot{r} + c)\dot{\theta} = 0 \tag{21}$$

(20) and (21) are the two well known equations of true proportional navigation |1|. The solution of this nonlinear system of differential equations will provide the trajectories of the missile in the relative system of coordinates previously defined.

3. CLOSED-FORM SOLUTION

In the classical theory of true proportional navigation it is tacitly assumed that the system of differential equations (20) and (21) is not solvable in closed form. Moreover, it is admitted, without proof, that the missile follows a straight line trajectory towards the pursuit end. The analysis that followed only considered small perturbations with respect to this final straight line trajectory.

In this study it will be shown that in fact there exists a closed form solution for system (20), (21) and this closed form solution will provide us with the conditions under which the missile captures the target.

Replacing a_M from (3) into (17)

$$\dot{V}_\theta = -(c+V_r)\,\dot{\theta} \tag{22}$$

$$\dot{V}_r = V_\theta\,\dot{\theta} \tag{23}$$

Multiplying (22) by V_θ and (23) by $(c + V_r)$ respectively and adding them

$$\dot{V}_\theta V_\theta + \dot{V}_r\,(c+V_r) = 0 \tag{24}$$

Rearranging

$$\frac{1}{2}\frac{d}{dt}(V_r^2+V_\theta^2) + c\,\frac{dV_r}{dt} = 0 \tag{25}$$

Integrating

$$V_\theta^2 + V_r^2 + 2cV_r = a \tag{26}$$

where

$$a = V_{\theta_0}^2 + V_{r_0}^2 + 2cV_{r_0} \quad . \tag{27}$$

Multiplying (23) by r

$$r\dot{V}_r = V_\theta^2 \quad . \tag{28}$$

Substituting V_θ^2 from (28) into (26)

$$r\dot{V}_r + V_r^2 + 2cV_r = a \quad , \tag{29}$$

Substituting V_r by $\dot{r}$ into (29)

$$r\ddot{r} + \dot{r}^2 + 2c\dot{r} = a \quad , \tag{30}$$

Equation (30) is an equation in r only. At this stage r and θ are separated.

Let differentiate $r\dot{r}$ with respect to time

$$\frac{d}{dt}(r\dot{r}) = r\ddot{r} + \dot{r}^2 \quad , \tag{31}$$

Replacing (31) into (30)

$$\frac{d}{dt}(r\dot{r}) + 2c\frac{dr}{dt} = a \quad . \tag{32}$$

Integrating

$$r\dot{r}+2cr=at+b \tag{33}$$

where

$$b=r_0\dot{r}_0+2cr_0 . \tag{34}$$

Let now

$$r=y+mt+n \tag{35}$$

where y is the new independent variable and m and n are two real constants

Substituting (35) into (33)

$$(y+mt+n)\dot{y}+(2c+m)y+(m+2c)mt+(m+2c)n=at+b \tag{36}$$

Let m and n be such that

$$(m+2c)m=a \tag{37}$$

and

$$n=mb/a \tag{38}$$

With these values of m and n (37) reduces to

$$(y+mt+n)\frac{dy}{dt}+ky=0 \tag{39}$$

where

$$k=m+2c \tag{40}$$

Before we proceed further it is important to remark that (37) has two solutions for m

$$m_1 = -c + \sqrt{c^2+a} \tag{41}$$

and

$$m_2 = -c - \sqrt{c^2+a} \tag{42}$$

Given the fact that (33) fulfills Cauchy Lipschitz condition for any real t and $r \neq 0$ it is sufficient to consider the solution for only one of the values of m. The other value will provide the same solution for r. Let in consequence $m = m_1$.

(39) is a homogeneous equation |4| and the variables can be separated. This equation is solved in Appendix I and the solution for y is

$$\left(\frac{y}{y_0}\right)^{m/k}\left[\frac{y+(m+k)x}{y_0+(m+k)x_0}\right] = 1 \tag{43}$$

where

$$x = t + n/m \tag{44}$$

and y_0 and x_0 are the initial values of y and x respectively and are obtained from (35) and (44) for $t = 0$

$$x_0 = n/m = b/a \tag{45}$$

$$y_0 = r_0 - n \tag{46}$$

Once the solution for y is obtained, r is obtained as shown in Appendix II.

The result is

$$r = r_0 \left(\mu_1 z^{-m/k} + \mu_2 z \right) \tag{47}$$

where $$z = y/y_0 \tag{48}$$

is defined by

$$p z^{-m/k} - q z = x \tag{49}$$

Note that $$z(x_0) = z_0 = y_0/y_0 = 1 \tag{50}$$

μ_1, μ_2, p and q are all real constants respectively defined by

$$\mu_1 = \frac{\nu + \sqrt{\nu^2+1}}{2\sqrt{\nu^2+1}} \tag{51}$$

$$\mu_2 = \frac{-\nu + \sqrt{\nu^2+1}}{2\sqrt{\nu^2+1}} \tag{52}$$

where

$$\nu = \frac{V_{r0} + c}{|V_{\theta 0}|} \tag{53}$$

and

$$p = \frac{y_0}{m+k} + x_0 = \frac{r_0 \mu_1}{m} \tag{54}$$

and

$$g = \frac{\gamma_2}{m+k} = \frac{r_0 \mu_2}{k} \tag{55}$$

Note that from (51) and (52)

$$0 \leq \mu_1 \leq 1 \tag{56}$$

$$0 \leq \mu_2 \leq 1 \tag{57}$$

Once the solution for r is obtained, $\dot{\theta}$ and θ are obtained as shown in Appendix III.

The result is

$$\dot{\theta} = \dot{\theta}_0 \frac{z^{(3m/k+1)2}}{(\mu_1 + \mu_2 z^{m/k+1})^2} \tag{58}$$

$$\theta = \theta_f - 2\,\mathrm{sign}(\dot{\theta}_0)\,\mathrm{arctg}\left(\frac{\mu_2}{\mu_1} z^{m/k+1}\right)^{1/2} \tag{59}$$

where

$$\theta_f = \theta_0 + 2\,\mathrm{sign}(\dot{\theta}_0)\,\mathrm{arctg}\left(\frac{\mu_2}{\mu_1}\right)^{1/2} \tag{60}$$

4. ANALYSIS OF THE SOLUTION

In order to analize the solution, the signs of a and b will first be determined.

From the expression for a (27), it follows that if

1) $(V_{r_0}, V_{\theta_0}) \in$ to a circle C_c defined by

$$(V_{r_0}+c)^2 + V_{\theta_0}^2 = c^2$$

with center at $(-c, 0)$ and radius c, then

$$a<0$$

2) $(V_{r_0}, V_{\theta_0}) \in$ to the circumference of C_c

$$a=0$$

3) $(V_{r_0}, V_{\theta_0}) \notin C_c$

$$a>0$$

From the definition of b (34):

1) $b<0$ for $V_{r_0}<-2c$

2) $b>0$ for $V_{r_0}>-2c$

In Fig. 3 we have depicted, in the plane V_{r_0}, V_{θ_0}, the circle C_c and the straight line $V_{r_0}=-2c$.

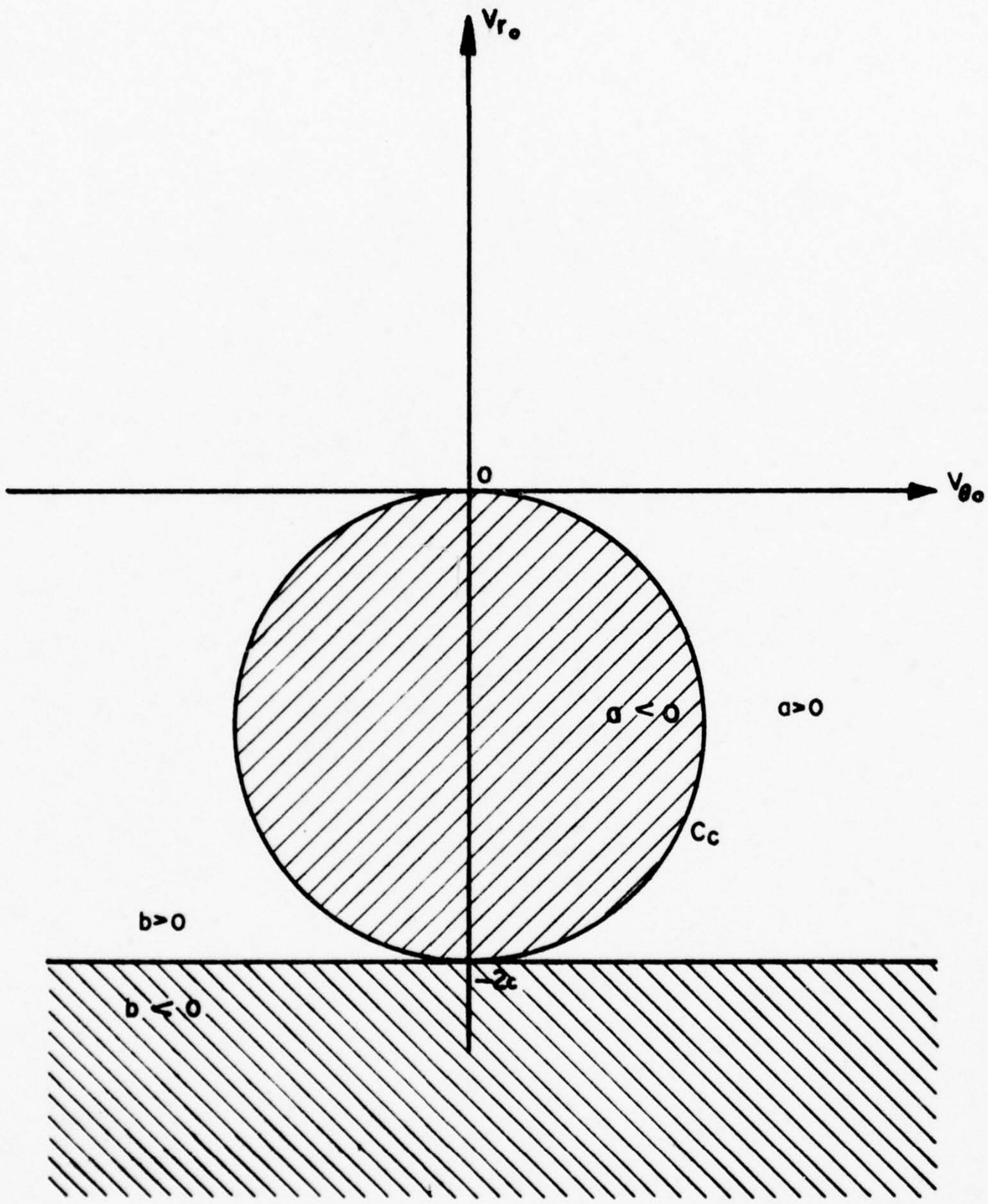

Fig. 3: The signs of a and b

Representing now the above results in table form:

Case A	$a > 0,\ b > 0$	$(V_{r_0}, V_{\theta_0}) \notin C_c,\ V_{r_0} > -2c$
Case B	$a < 0,\ b > 0$	$V_{r_0} \in C_c$
Case C	$a > 0,\ b < 0$	$V_{r_0} < -2c$
Case D	$a < 0,\ b < 0$	

Case A) $a > 0,\ b > 0$

Rearranging equation (33)

$$r(\dot{r} + 2c) = at + b \tag{61}$$

From (61) it follows that a necessary condition for the missile M to reach the target T ($r = 0$) is

$$at + b = 0 \quad \text{for} \quad t = t_1 \quad , \ 0 < t_1 < \infty$$

On the other hand

$$at + b = 0 \quad \text{for} \quad t = t_1$$

if and only if

$$a.b < 0 \tag{62}$$

<u>THEOREM 1</u>: A missile M pursuing a nonmaneuvering target T according to the true proportional navigation law and starting its course at $M = M_0(r_0, \theta_0)$ where

$$(V_{r_0}, V_{\theta_0}) \notin C_c \ , \quad V_{r_0} > -2c$$

will not reach the target for any real t.

Case B) $a < 0$, $b > 0$.

From (41), with $a < 0$ it follows

$$m = m_1 = -c + \sqrt{c^2 + a} < 0 \tag{63}$$

Substituting this value of m into (40)

$$k = m + 2c = c + \sqrt{c^2 + a} > 0 \tag{64}$$

and

$$m + k = 2\sqrt{c^2 + a} > 0 \tag{65}$$

It follows then

$$0 < -m/k < 1 \tag{66}$$

and

$$0 < (m+k)/k < 1 \tag{67}$$

Now, from (54) and (55) we have

$$p < 0 \tag{68}$$

$$q > 0 \tag{69}$$

and from (55), (67) and (57)

$$0 < y_0 = \tau_0 \frac{(m+k)}{k} \mu_2 < \tau_0 \tag{70}$$

We shall first study z as a function of x as defined in (49)

For $t = 0$, $x = x_0 = b/a$, and $z_0 = 1$.

For $x = 0$ it follows $z = 0$.

Differentiating (49) with respect to x and rearranging

$$\frac{dz}{dx} = -k \frac{z^{m/k+1}}{r_0(\mu_1 + \mu_2 z^{m/k+1})} \tag{71}$$

hence with $k > 0$

$$\frac{dz}{dx} < 0 \tag{72}$$

and from(66), $m/k + 1 > 0$, thus

$$\lim_{z \to 0} \frac{dz}{dx} = 0 \tag{73}$$

With all these elements z is depicted as a function of x in Fig. 4. y as a function of x is directly obtained multiplying z by y_0 as depicted in Fig. 4. From (44) it follows that y as a function of t is obtained translating the origin along the x axis by $n/m = b/a$. This is depicted in Fig. 5. Finally, recalling (35), r is obtained by adding to y, $mt + n$. This is also depicted in Fig. 5.

It results in consequence that $r = 0$ for $t = -b/a$. The missile reaches the target in this case.

The value of t

$$t = t_f = -\frac{b}{a} = \left(-\frac{r_0}{V_{r_0}}\right)\left[1 - \frac{V_{\theta_0}^2}{V_{r_0}(V_{r_0} + 2c) + V_{\theta_0}^2}\right] \tag{74}$$

is the final time of the pursuit.

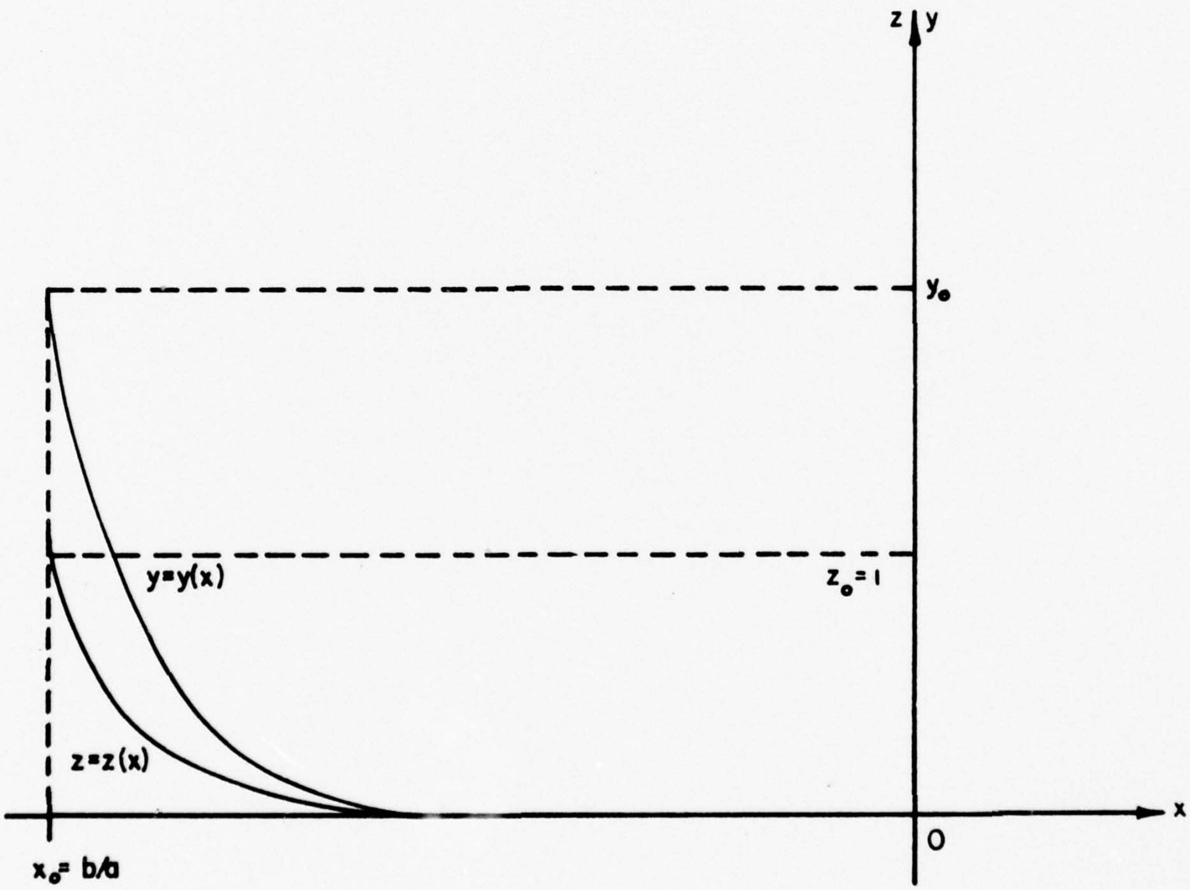

Fig. 6: y and z vs. x

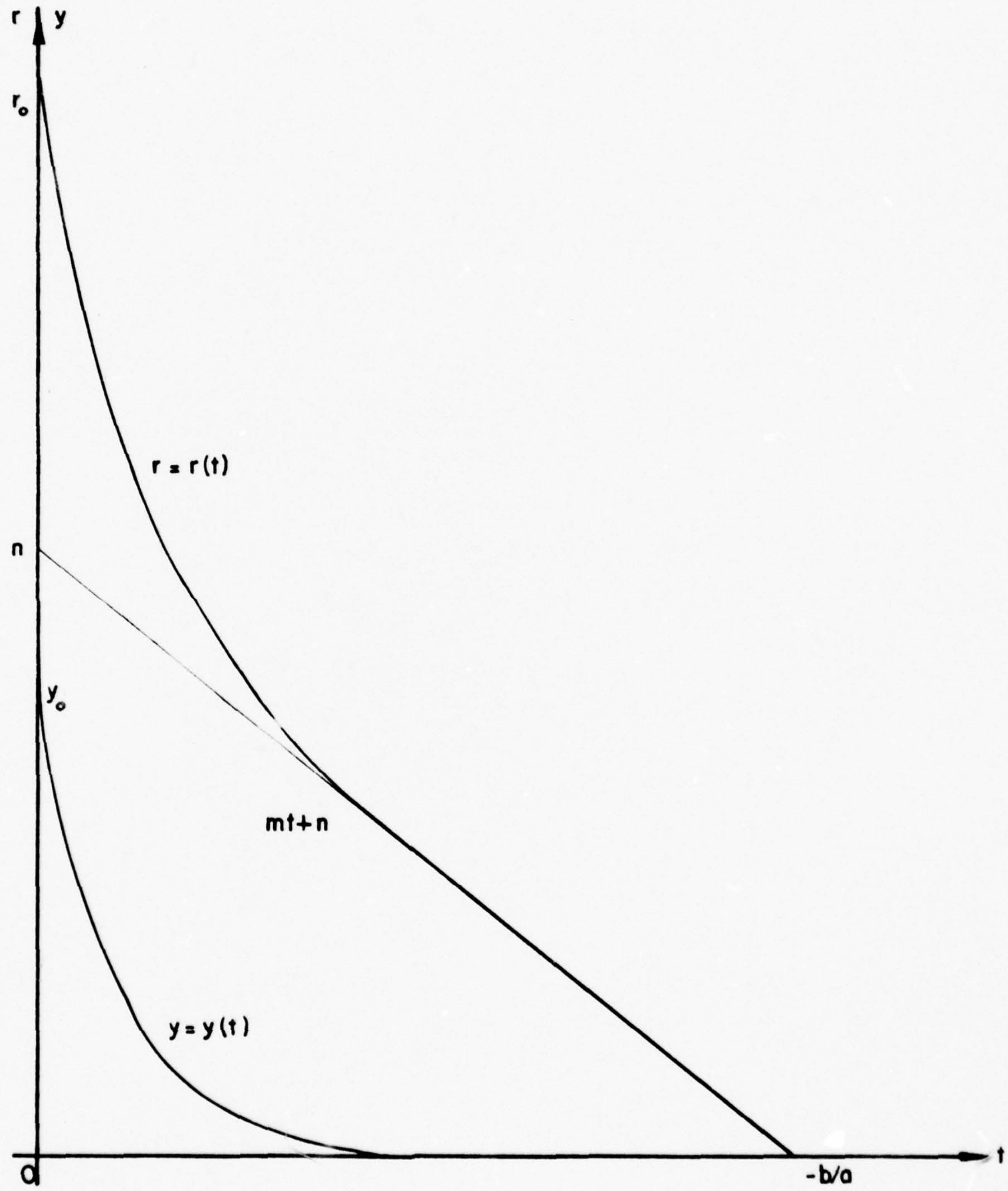

Fig. 5: r and y vs. time t

It is worth while to remark that when $V_{\theta_0}^2 = 0$, $t_f = -\frac{r_0}{V_{r_0}}$ as we should expect.

In what concerns the closing velocity, differentiating (35) with respect to time t.

$$\dot{r} = \dot{y} + m \tag{75}$$

Now, from (44) and (48) it follows

$$\frac{dy}{dt} = \frac{dy}{dx} = y_0 \frac{dz}{dx} \tag{76}$$

From (73) for $z = 0$ $(t = t_f)$

$$\frac{dy}{dt}(t_f) = 0 \tag{77}$$

Hence

$$V_{r_f} = \dot{r}_f = m = -c + \sqrt{(V_{r_0} + c)^2 + V_{\theta_0}^2} \tag{78}$$

The angle θ is obtained from (59). For $z = 0$.

$$\theta = \theta_f = \theta_0 + 2\,\mathrm{sign}(\dot{\alpha}_0)\,\mathrm{arctg}\left(\frac{\mu_2}{\mu_1}\right)^{1/2} \tag{79}$$

THEOREM 2: A missile M pursuing a nonmaneuvering target T according to true proportional navigation, starting at $M_0(r_0, \theta_0)$ such that

$$(V_{r_0}, V_{\theta_0}) \in C_c \tag{80}$$

reaches the target at a finite time

$$t = t_f = \left(\frac{-r_0}{V_{r_0}}\right)\left[1 - \frac{V_{\theta_0}^2}{V_{r_0}(V_{r_0}+2c) + V_{\theta_0}^2}\right] \tag{81}$$

Moreover M arrives to T with a closing speed

$$V_r = V_{r_f} = -c + \sqrt{(V_{r_0}+c)^2 + V_{\theta_0}^2} \tag{82}$$

at an aspect angle

$$\theta = \theta_f = \theta_0 + 2\,\mathrm{sign}(\dot{\theta}_0)\,\mathrm{arctg}\left[\frac{-(V_{r_0}+c) + \sqrt{(V_{r_0}+c)^2 + V_{\theta_0}^2}}{(V_{r_0}+c) + \sqrt{(V_{r_0}+c)^2 + V_{\theta_0}^2}}\right]^{1/2} \tag{83}$$

Remark 1: The conditions to capture the target depend on the initial conditions and their relations with c.

No conditions at all are imposed on $v = V_M/V_T$, the ratio of velocities as was the case in pure proportional navigation. Even when $v < 1$ capture is possible.

The rate of rotation of the line of sight plays a fundamental role in missile design. For a stable functioning of the missile it is essential that $\dot{\theta}$ should be bounded.

From the expression of $\dot{\theta}$, (58)

$$\dot{\theta} = \dot{\theta}_0 \frac{z^{(3m/k+1)/2}}{(\mu_1 + \mu_2 z^{m/k+1})^2}$$

it follows that if $k > 0$ and

1) $3m + k < 0$ (84)

then $\lim_{z \to 0} |\dot{\theta}| = \infty$

2) $3m + k = 0$ (85)

$$\lim_{z \to 0} \dot{\theta} = \dot{\theta}_0 / \mu_1^2$$

and

3) $3m + k > 0$ (86)

$$\lim_{z \to 0} \dot{\theta} = 0$$

Substituting for m and k their values given in (63) and (64) into (84), (85) and (86) and rearranging we obtain respectively that if

1) $(V_{r_0}, V_{\theta_0}) \in C_s$, where C_s is a circle defined by

$$(V_{r_0} + c)^2 + V_{\theta_0}^2 = (c/2)^2$$

then $\lim_{t \to t_f} |\dot{\theta}| = \infty$

2) $(V_{r_0}, V_{\theta_0}) \in$ to the circumference of C_s

$$\lim_{t \to t_f} \dot{\theta} = \dot{\theta}_0 / \mu_1^2$$

3) $(V_{r_0}, V_{\theta_0}) \notin C_s$

$$\lim_{t \to t_f} \dot{\theta} = 0$$

Let us determine now the value of $\ddot{\theta}$. Rearranging (21)

$$\ddot{\theta} = -\left(\frac{c+2\dot{r}}{r}\right)\dot{\theta} \tag{87}$$

From (75) and(76) it follows

$$\dot{r} = y_0 \frac{dz}{dx} + m \tag{88}$$

Substituting $\frac{dz}{dx}$ defined in (71) into (88)

$$\dot{r} = -y_0 \frac{k z^{m/k+1}}{r_0(\mu_1+\mu_2 z^{m/k+1})} + m \tag{89}$$

Substituting r, ṙ and $\dot{\theta}$ from (47), (89) and (58) into (87) and rearranging

$$\ddot{\theta} = -\dot{\theta}_0 \frac{z^{(5m/k+1)/2}\left[\mu_1(c+2m)+\mu_2(c-2k)z^{m/k+1}\right]}{r_0(\mu_1+\mu_2 z^{m/k+1})^4} \tag{90}$$

Now

$$c+2m = 3\frac{m+k}{2} \tag{91}$$

thus, c + 2m > 0 if $(V_{r_0}, V_{\theta_0}) \notin C_S$ and c + 2m < 0 for $(V_{r_0}, V_{\theta_0}) \in C_S$.

$$c-2k = -c-2\sqrt{c^2+a} < 0 \tag{92}$$

For c + 2m > 0 , there exists z

$$z = z_1 = \left[-\frac{\mu_1(c+2m)}{\mu_2(c-2k)}\right]^{\frac{k}{m+k}} \tag{93}$$

such that if

$$z_1 < 1$$

then $\ddot{\theta}(t_1) = 0$ for $0 < t_1 < t_f$, where t_1 is such that $z(t_1) = z_1$.

Substituting μ_1, μ_2, m and k into z_1 it is readily shown that $z_1 < 1$ if

$$V_{r_0} < -c/2 \tag{94}$$

Moreover, for $z > z_1$ $(t < t_1)$

$$\mathrm{Sign}(\ddot{\theta}) = \mathrm{Sign}(\dot{\theta}_0) \tag{95}$$

and for $z < z_1$ $(t > t_1)$

$$\mathrm{Sign}(\ddot{\theta}) = -\mathrm{Sign}(\dot{\theta}_0) \tag{96}$$

It follows then, that for $\dot{\theta}_0 > 0$ $(\dot{\theta}_0 < 0)$ $\dot{\theta}(t)$ has a maximum (minimum) at $t = t_1$.

In what concerns the value of $\ddot{\theta}$ at $t = 0$, it is directly obtained from (87)

$$\ddot{\theta}_0 = -(c + 2V_{r_0})\left(\frac{\dot{\theta}_0}{V_{r_0}}\right) \tag{97}$$

Thus, if $V_{r_0} > -c/2$,

$$\mathrm{Sign}(\ddot{\theta}_0) = -\mathrm{Sign}(\dot{\theta}_0) \tag{98}$$

and if $V_{r_0} < -c/2$

$$\mathrm{Sign}(\ddot{\theta}_0) = \mathrm{Sign}(\dot{\theta}_0) \tag{99}$$

For $t = t_f$, if

1) $5m + k < 0$ (100)

$$\lim_{z \to 0} |\ddot{\theta}| = \infty$$

2) $5m + k = 0$ (101)

$$\lim_{z\to 0} \ddot{\theta} = -\frac{\dot{\theta}_0}{r_0}\frac{(c+2m)}{\mu_1^3}$$

3) $5m + k > 0$ (102)

$$\lim_{z\to 0} \ddot{\theta} = 0$$

Substituting m and k into (100), (101) and (102) we obtain, if

1) $(V_{r_0}, V_{\theta_0}) \in C_D$ where C_D is a circle defined by

$$(V_{r_0}+c)^2 + V_{\theta_0}^2 = (2c/3)^2$$

then

$$\lim_{t\to t_f} |\ddot{\theta}| = \infty$$

2) $(V_{r_0}, V_{\theta_0}) \in$ to the circumference of C_D

$$\lim_{t\to t_f} \ddot{\theta} = -\frac{\dot{\theta}_0}{r_0}\frac{(c+2m)}{\mu_1^3}$$

3) $(V_{r_0}, V_{\theta_0}) \notin C_D$

$$\lim_{t\to t_f} \ddot{\theta} = 0$$

In Fig. 6 all the three circles C_c, C_s and C_D and the straight line $V_{r_0} = -c/2$ are depicted.

For the case 1), $(V_{r_0}, V_{\theta_0}) \in C_D$, previously considered we can distinguish between three subcases

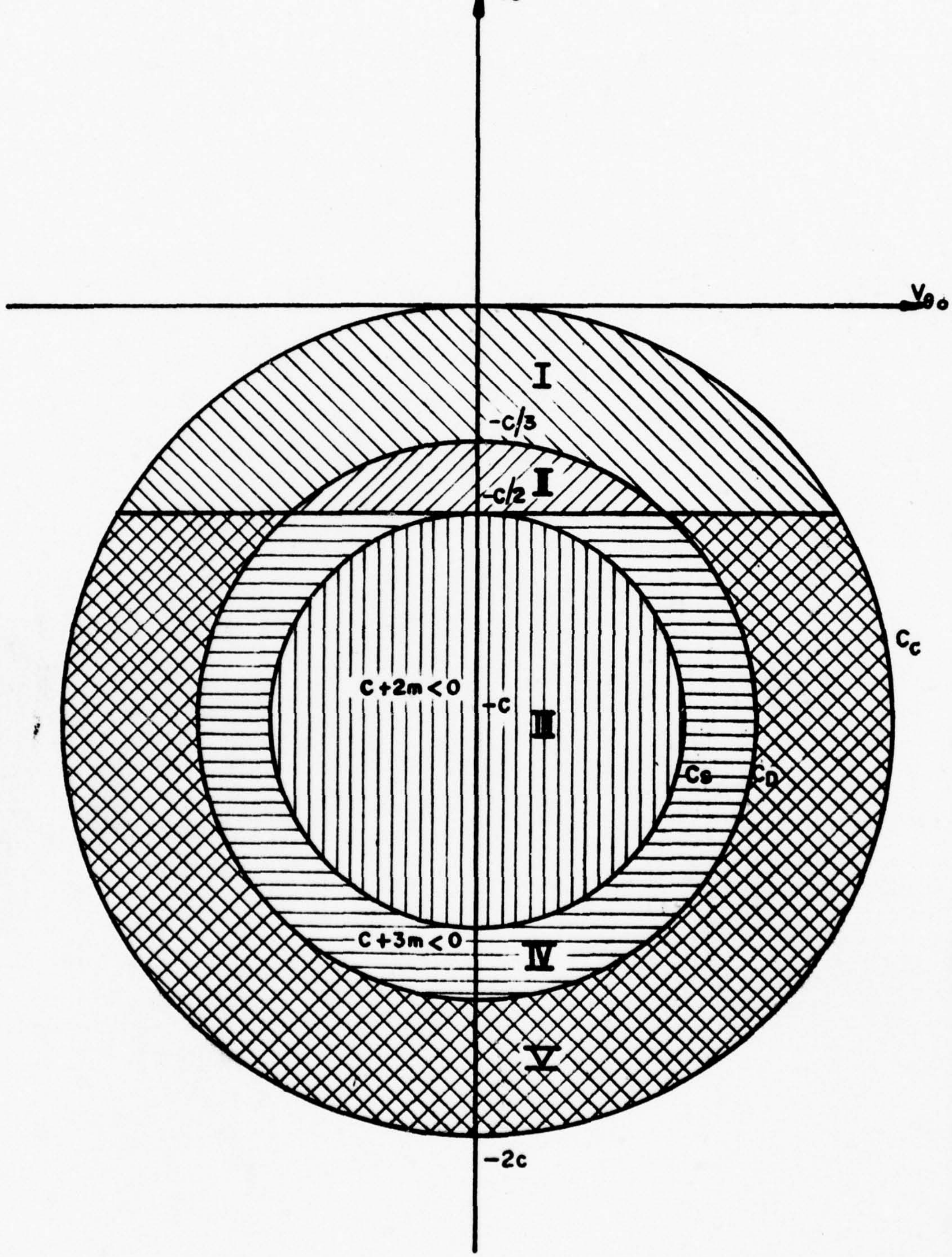

Fig. 6: The zones I to V

1.1) $(V_{r_0}, V_{\theta_0}) \in C_s$

$$\lim_{t \to t_f} \ddot{\theta} = [\mathrm{Sign}(\dot{\theta}_0)] \infty$$

1.2) $(V_{r_0}, V_{\theta_0}) \in$ to the circumference of C_s

$$\lim_{t \to t_f} \ddot{\theta} = 0$$

1.3) $(V_{r_0}, V_{\theta_0}) \notin C_s$

$$\lim_{t \to t_f} \ddot{\theta} = -[\mathrm{Sign}(\dot{\theta}_0)] \infty$$

We can distinguish now five different zones, denoted I to V.

I) $V_{r_0} > -c/2$, $(V_{r_0}, V_{\theta_0}) \notin C_D$, $\in C_c$

$$\dot{\theta}(t_f) = 0, \quad \ddot{\theta}(t_f) = 0$$

$$\mathrm{Sign}(\ddot{\theta}) = -\mathrm{Sign}(\dot{\theta}_0)$$

II) $V_{r_0} > -c/2$, $(V_{r_0}, V_{\theta_0}) \in C_D$, $\in C_c$

$$\dot{\theta}(t_f) = 0, \quad \ddot{\theta}(t_f) = -[\mathrm{Sign}(\dot{\theta}_0)] \infty$$

$$\mathrm{Sign}(\ddot{\theta}) = -\mathrm{Sign}(\dot{\theta}_0)$$

III) $V_{r_0} \in C_S$; $\in C_c$

$$\dot{\theta}(t_f) = \infty, \quad \ddot{\theta}(t_f) = [\mathrm{Sign}(\dot{\theta}_0)] \infty$$

$$\mathrm{Sign}(\ddot{\theta}) = \mathrm{Sign}(\dot{\theta}_0)$$

IV) $V_{r_0} < -c/2$, $(V_{r_0}, V_{\theta_0}) \notin C_S$, $\in C_D$, $\in C_c$

$$\dot{\theta}(t_f) = 0, \quad \ddot{\theta}(t_f) = -[\mathrm{Sign}(\dot{\theta}_0)]\infty$$

$\dot{\theta}$ has an extremum (maximum for $\dot{\theta}_0 > 0$) for $t = t_1$

$$\mathrm{Sign}(\ddot{\theta}_0) = \mathrm{Sign}(\dot{\theta}_0).$$

V) $V_{r_0} < -c/2$, $(V_{r_0}, V_{\theta_0}) \in C_D$, $\in C_c$

$$\dot{\theta}(t_f) = 0, \quad \ddot{\theta}(t_f) = 0$$

$\dot{\theta}$ has an extremum for $t = t_1$

$$\mathrm{Sign}(\ddot{\theta}_0) = \mathrm{Sign}(\dot{\theta}_0).$$

Representing now $\dot{\theta}$ as a function of t we obtain the five different curves depicted in Fig. 7.

<u>THEOREM 3</u>: The commanded acceleration of a missile M pursuing a nonmaneuvering target T according to true proportional navigation

$$a_M = c\dot{\theta}$$

for M starting its course at M_o such that

$$(V_{r_0}, V_{\theta_0}) \notin C_S, \in C_c$$

is a bounded function of time (zones I, II, IV, V).

For

$$(V_{r_0}, V_{\theta_0}) \in C_S$$

(zone III) a_M becomes unbounded at the pursuit end.

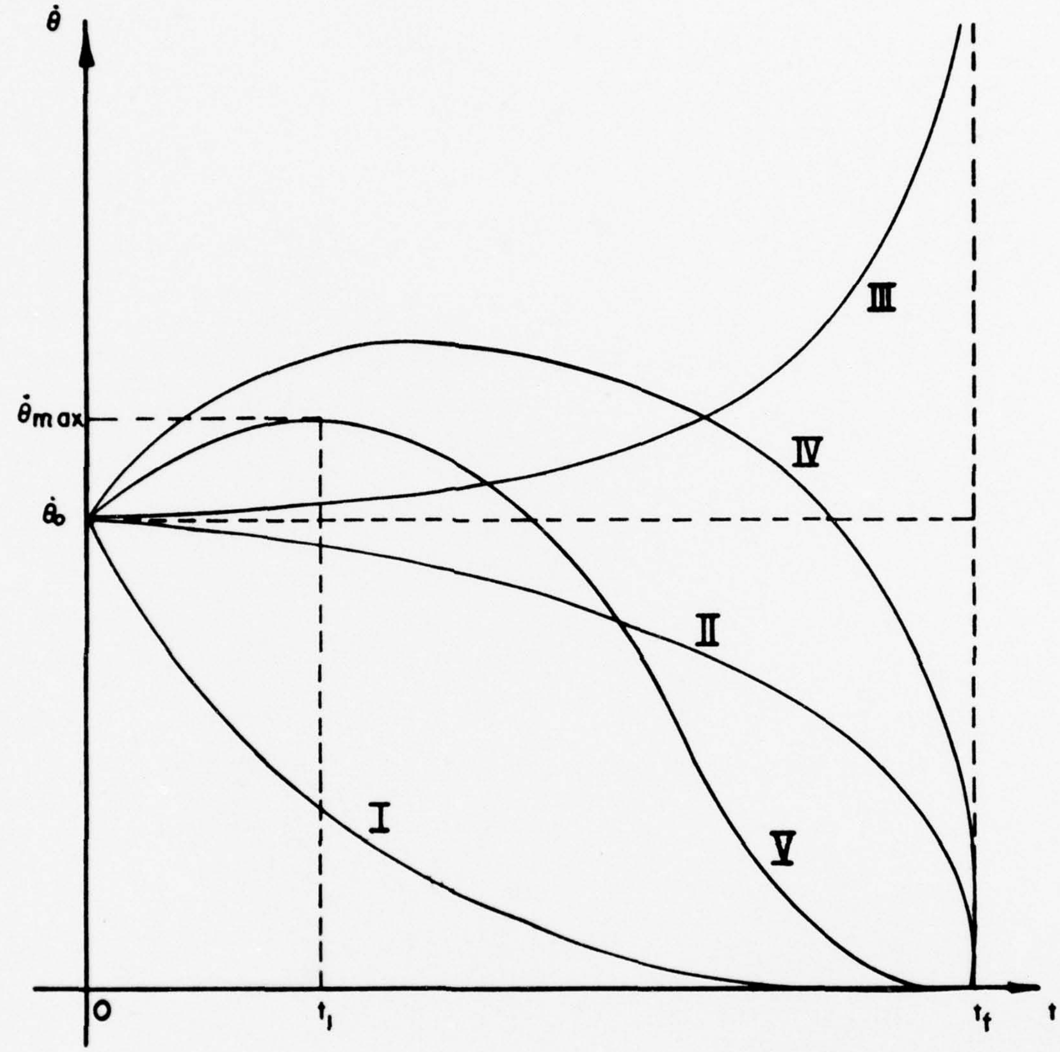

Fig. 7: Line of sight rotation rate vs. time for (V_{r_0}, V_{θ_0}) belonging to zones I to V.

Case C) $a > 0$, $b < 0$.

For this case

$$m > 0 \tag{103}$$

and

$$k > 0 \tag{104}$$

From the expression (47) of r it is readily seen that for $m/k > 0$ there does not exist any real t for which

$$r = 0 \tag{105}$$

In consequence it is obtained the following result

THEOREM 4: A missile M pursuing a nonmaneuvering target T according to true proportional navigation, starting at M_0 where

$$V_{r_0} < -2c \tag{106}$$

will not reach the target for any real t.

5. A PARTICULAR CASE

The system of differential equations (20), (21) has the particular solution

$$\dot{\theta} = 0 \tag{107}$$

$$\ddot{r} = 0 \tag{108}$$

as can be directly proved by substitution.

In terms of V_r and V_θ this solution corresponds to the case

$$V_\theta = 0 \tag{109}$$

$$V_r = cte. \tag{110}$$

In the plane V_r, V_θ depicted in Fig. 8, this particular case corresponds to the points belonging to the straight line $V_\theta = 0$.

For $V_r < 0$ the missile reaches the target without maneuvering

$$a_M = c\dot{\theta} = 0 \tag{111}$$

In previous works |1| the analysis of true proportional navigation was restricted to the neighbourhood of this particular case

$$V_\theta \simeq 0 \tag{112}$$

$$V_r \simeq cte. \tag{113}$$

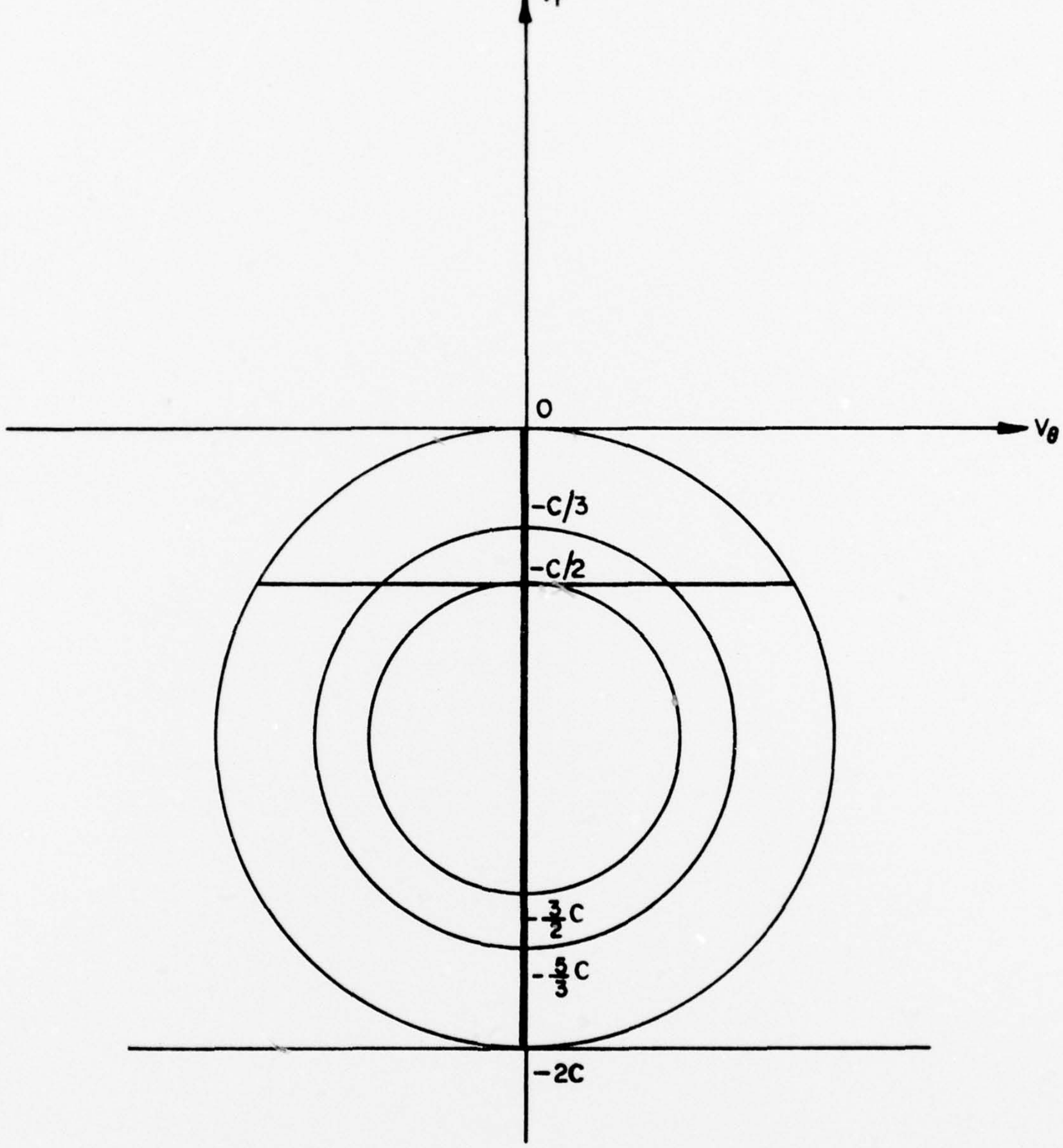

Fig. 8: The case $V_\theta = 0$, $V_r =$ ct.

In this case, as previously mentioned, c is defined as

$$c = -\lambda V_r \tag{114}$$

where λ is the navigation constant.

It follows from the results of the previous section that the missile reaches the target for $V_\theta \simeq 0$ if

$$V_r < 0 \tag{115}$$

and

$$V_T > -2c = 2\lambda V_r \tag{116}$$

that is

$$\lambda > 1/2 \tag{117}$$

In what concerns the rate of rotation of the line of sight we can distinguish between five different cases

I) $3 < \lambda$

II) $2 < \lambda < 3$

III) $2/3 < \lambda < 2$

IV) $3/5 < \lambda < 2/3$

V) $1/2 < \lambda < 3/5$

These five different cases are depicted in Fig. 7.

SUMMARY AND CONCLUSIONS

In this study it was derived the closed form solution of the differential equations describing the trajectories of a missile pursuing a nonmaneuvering target according to the true proportional navigation law.

The solution was analized and a circle was defined where capture can be demonstrated. For the case of initial conditions belonging to the circle of capture the rate of rotation of the line of sight was analised and new results were found concerning its boundedness at the pursuit end.

The point of greatest interest in this study is the fact that the analysis of the closed-form solution of TPN enabled to demonstrate the basic differences existing between the two most utilised forms of proportional navigation. Essentially, when in PPN capture of the target can be assured for the entire plane of initial conditions, excepted a well defined particular case, in TPN capture is restricted to the here defined circle of capture.

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APPENDIX I

Solution of

$$(y+mt+n)\frac{dy}{dt}+ky=0 \tag{118}$$

Let

$$t=x-n/m \tag{119}$$

Substituting (119) into (118) and rearranging

$$\frac{dy}{dx}=-\frac{ky}{y+mx} \tag{120}$$

Defining a new variable u instead of y

$$y=ux \tag{121}$$

Differentiating (121) with respect to x

$$\frac{dy}{dx}=u+x\frac{du}{dx} \tag{122}$$

Substituting (121) and (122) into (120)

$$x\frac{du}{dx}+u=-\frac{ku}{u+m} \tag{123}$$

Rearranging

$$\frac{dx}{x} = \frac{-(u+m)}{[u+(m+k)]u}\,du \tag{124}$$

Integrating (124) with initial conditions $u = u_0$, $x(t=0) = x_0 = n/m$

$$\log\left[\left(\frac{u}{u_0}\right)^{\frac{-m}{m+k}}\left(\frac{u+m+k}{u_0+m+k}\right)^{\frac{-k}{m+k}}\right] = \log\frac{x}{x_0} \tag{125}$$

thus

$$\left(\frac{u}{u_0}\right)^{\frac{-m}{m+k}}\left(\frac{u+m+k}{u_0+m+k}\right)^{\frac{-k}{m+k}} = \frac{x}{x_0} \tag{126}$$

Substituting u from (121) into (126) and rearranging, with $y_0 = u_0 x_0$

$$\left(\frac{x}{x_0}\right)\left(\frac{y}{y_0}\right)^{\frac{-m}{m+k}}\left(\frac{y+(m+k)x}{y_0+(m+k)x_0}\right)^{\frac{-k}{m+k}} = \frac{x}{x_0} \tag{127}$$

Eliminating x/x_0 and elevating to $-(m+k)/k$

$$\left(\frac{y}{y_0}\right)^{m/k}\left(\frac{y+(m+k)x}{y_0+(m+k)x_0}\right) = 1 \tag{128}$$

APPENDIX II

The solution for r is obtained as follows:

Let

$$z = y/y_0 \tag{129}$$

Rearranging (128) and substituting y by z

$$x = pz^{-m/k} - qz \tag{130}$$

where

$$p = \frac{y_0}{m+k} + x_0 \tag{131}$$

and

$$q = \frac{y_0}{m+k} \tag{132}$$

Substituting x from (130) into (119), and the corresponding value of t so obtained into (35)

$$r = mpz^{-m/k} + qkz \tag{133}$$

where $kq = y_0 - mq$.

From (27) and (41)

$$m = -c + \sqrt{(V_{r_0}+c)^2 + V_{\theta_0}^2} \tag{134}$$

thus, from (40)

$$k = c + \sqrt{(V_{r_0}+c)^2 + V_{\theta_0}^2} \tag{135}$$

and

$$m+k = 2\sqrt{(V_{r_0}+c)^2 + V_{\theta_0}^2} \tag{136}$$

Substituting now x_0 from (45) and y_0 from (46) into (131) and (132) and rearranging

$$p = \frac{r_0 m + nk}{m(m+k)} \tag{137}$$

and

$$q = \frac{r_0 - n}{m+k} \tag{138}$$

Substituting k from (40) into (37) and the value of a so obtained into (38) it follows

$$nk = b \tag{139}$$

Multiplying (131) by m and replacing nk from (139)

$$mp = \frac{m r_0 + b}{m+k} \tag{140}$$

Substituting now m, b and m + k by their values given in (134), (34) and (136) respectively and rearranging

$$mp = r_0 \frac{\left[(V_{r_0}+c) + \sqrt{(V_{r_0}+c)^2 + V_{\theta_0}^2}\right]}{2\sqrt{(V_{r_0}+c)^2 + V_{\theta_0}^2}} \tag{141}$$

Let us define

$$\nu = \frac{V_{r_0} + c}{|V_{\theta_0}|} \tag{142}$$

Substituting into (141)

$$mp = r_0 \mu_1 \tag{143}$$

where

$$\mu_1 = \frac{\nu + \sqrt{\nu^2 + 1}}{2\sqrt{\nu^2 + 1}} \tag{144}$$

Substituting now (139) into (138) and rearranging

$$qk = \frac{kr_0 - b}{m + k} \tag{145}$$

from where it follows

$$kq = r_0 \mu_2 \tag{146}$$

where

$$\mu_2 = \frac{-\nu + \sqrt{\nu^2 + 1}}{2\sqrt{\nu^2 + 1}} \tag{147}$$

Substituting now mp and qk from (143) and (146) respectively into (138)

$$r = r_0 \left(\mu_1 z^{-m/k} + \mu_2 z\right) \tag{148}$$

This is the solution for r as a function of z. z is implicitty defined in (130) with x defined in (119).

APPENDIX III

The solution for $\dot{\theta}$, the rate of rotation of the line of sight, and θ the aspect angle, is obtained as follows:

$$\frac{dy}{dt} = \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} \tag{149}$$

Differentiating (129)

$$\frac{dy}{dt} = y_0 \frac{dz}{dx} \tag{150}$$

Differentiating once again

$$\frac{d^2y}{dt^2} = y_0 \frac{d^2z}{dx^2} \tag{151}$$

Differentiating now (35) twice with respect to t

$$\frac{d^2r}{dt^2} = \frac{d^2y}{dt^2} \tag{152}$$

Substituting (151) into (152)

$$\ddot{r} = y_0 \frac{d^2z}{dx^2} \tag{153}$$

Differentiating now (130) with respect to x and rearranging

$$\frac{dz}{dx} = \frac{-k z^{m/k+1}}{r_0 (\mu_1 + \mu_2 z^{m/k+1})} \tag{154}$$

Differentiating once again with respect to x and operating

$$\frac{d^2z}{dx^2} = \frac{d}{dz}\left(\frac{dz}{dx}\right)\cdot\frac{dz}{dx} = \frac{k(m+k)\mu_1 \, z^{2m/k+1}}{r_0^2(\mu_1+\mu_2 z^{m/k+1})^3} \tag{155}$$

Substituting (155) into (153)

$$\ddot{r} = \frac{V_{\theta_0}^2}{r_0^2}\,\frac{z^{2m/k+1}}{(\mu_1+\mu_2 z^{m/k+1})^3} \tag{156}$$

Rearranging (20)

$$\dot{\theta}^2 = \ddot{r}/r \tag{157}$$

Substituting r from (148) and $\ddot{r}$ from (156) respectively into (157) and rootsquaring

$$\dot{\theta} = \dot{\theta}_0\,\frac{z^{\frac{2m/k}{2}+1}}{(\mu_1+\mu_2 z^{m/k+1})^2} \tag{158}$$

This is the solution for the rate of rotation of the line of sight as a function of z defined in (130).

θ is obtained as follows

$$\theta-\theta_0 = \int_0^t \dot{\theta}(t)\,dt = \int_{x_0}^{x} \dot{\theta}(x)\,dx \tag{159}$$

Changing the variable x by z, where for $x = x_0$, $y = y_0$ and consequently $z = 1$

$$\theta-\theta_0 = \int_1^z \left[\dot{\theta}(z)\,\frac{dx}{dz}\right] dz \tag{160}$$

Rearranging (154)

$$\frac{dx}{dz} = -\frac{r_0(\mu_1 + \mu_2 z^{m/k+1})}{k z^{m/k+1}} \tag{161}$$

Substituting (158) and (161) into (160)

$$\Theta - \Theta_0 = -\frac{V_{\theta 0}}{k}\int_1^z \frac{z^{\frac{m/k-1}{2}}}{\mu_1 + \mu_2 z^{m/k+1}}\, dz \tag{162}$$

Let

$$s = z^{m/k+1} \tag{163}$$

thus

$$dz = \frac{k s^{-m/(m+k)}}{m+k}\, ds \tag{164}$$

Substituting into (162)

$$\Theta - \Theta_0 = -\frac{V_{\theta 0}}{m+k}\int_1^s \frac{s^{-1/2}}{\mu_1 + \mu_2 s}\, ds \tag{165}$$

Hence

$$\Theta - \Theta_0 = -2\,\mathrm{sign}(V_{\theta 0})\,\mathrm{arctg}\left(\frac{\mu_2}{\mu_1}s\right)^{1/2}\Bigg|_1^s \tag{166}$$

Substituting s by z as defined in (163)

$$\Theta = -2\,\mathrm{sign}(V_{\theta 0})\,\mathrm{arctg}\left(\frac{\mu_2}{\mu_1}z^{m/k+1}\right)^{1/2} + \Theta_f \tag{167}$$

where

$$\Theta_f = \Theta_0 + 2\,\mathrm{sign}(V_{\theta 0})\,\mathrm{arctg}\left(\frac{\mu_2}{\mu_1}\right)^{1/2} \tag{168}$$